Periodic Motions in Natural and Designed Systems

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Symplectic Geometry

A **symplectic manifold** is a $2n$-dimensional C^∞ manifold M with a closed non-degenerate 2-form ω, called the **symplectic form**.

Symplectic geometry originates from the Hamilton's formulation of classical mechanics [Arn13; Abr19] where the **phase space** takes on the structure of a *symplectic manifold*. Solutions to **Hamilton's equations** are integral curves of the **Hamiltonian vector field**.

Cotangent Bundle

If X is a n-dimensional C^∞ manifold, its **cotangent bundle** T^*X is equipped with the **canonical symplectic form** ω.[a] In the case of $X = \mathbb{R}^n$, then $T^*X = \mathbb{R}^{2n}$ and **the standard symplectic form** ω_0 is

$$\omega_0(p)(x,y) \triangleq \sum_1^n \det\left(\left[\begin{smallmatrix} x_i & y_i \\ x_{n+i} & y_{n+i} \end{smallmatrix}\right]\right), \quad \forall x,y \in T_p\mathbb{R}^{2n},\ p \in \mathbb{R}^{2n}.$$

If $H \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ its Hamiltonian vector field in coordinates is

$$X_H(p) = J_{2n}\nabla H(p), \quad \text{where} \quad J_{2n} \triangleq \begin{bmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ -\mathrm{id}_{\mathbb{R}^n} & 0 \end{bmatrix}, \quad p \in \mathbb{R}^{2n}.$$

Regular Hamiltonians

Let $(M,\ \omega)$ be a symplectic manifold. $H \in C^\infty(M)$ is **regular** if there is an open, non-empty $U \subseteq M$ and compact $K \subseteq M \setminus \partial M$ such that $0 \le H(x)$, $H(y) = 0$, and $H(z) = \max(H) < \infty$ for all $x \in M$, $y \in U$, and $z \notin K$

$$\mathcal{H}(M,\omega) \triangleq \{H \in C^\infty(M) \text{ is regular}\}.$$

The **magnitude** of $H \in \mathcal{H}(M,\omega)$ is $\mathrm{mag}(H) \triangleq \max(H)$ is a measure of the extremeness of a special class of Hamiltonians on M.

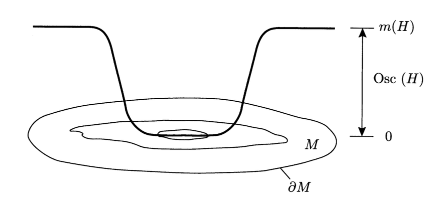
Admissible Hamiltonians

$H \in \mathcal{H}(M,\ \omega)$ is **admissible** if all periodic orbits[b] of $\dot{x} = X_H(x)$ have period $T > 1$ (if there are any).

$$\mathcal{H}_{ad}(M,\omega) \triangleq \{H \in C^\infty(M) \text{ is admissible}\}.$$

[a] If (X, g) is a **Riemannian manifold**, T^*X (with local coordinates $(q,p) \in T^*X$) serves as the natural setting for the Hamilton formalism with $H(q,p) = V(q) + T(q,p)$ where $T(q,p) \triangleq 2^{-1}g(q)(p,p)$ and $V \in C^\infty(X)$ under the tangent-cotangent identification [Sil04; Lan12; Lee13]. The term $T(q,p)$ is the **kinetic energy** of the system, while $V(q)$ is the **potential energy**. If $\partial_t H = 0$, then H = **total energy**.

[b] We use the term **orbit** only for **non-constant solutions**



Regular Hamiltonian, figure taken from [HZ12].

Symplectic Capacities

A **symplectic capacity** associates a symplectic manifold (M,ω) a number $\mathcal{C}(M,\omega) \in [0,\infty]$ satisfying

Monotonicity $\mathcal{C}(M,\omega) \le \mathcal{C}(N,\eta)$ if M embeds into N symplectically,

Conformality $(M, \alpha\omega) = |\alpha|(M,\omega)$ for all $\alpha \neq 0$, and

Non-triviality $\mathcal{C}(B(1),\ \omega_0) = \mathcal{C}(Z(1),\ \omega_0) = \pi$.[a]

Hofer-Zehnder Capacity

The **Hofer-Zehnder capacity** of a symplectic manifold (M,ω) is

$$\mathcal{C}_0(M,\ \omega) \triangleq \sup\{\mathrm{mag}(H),\ H \in \mathcal{H}_{ad}(M,\ \omega)\}.$$

From [HZ12; Jia93]: $\mathcal{C}_0$ is a symplectic capacity, and $\mathcal{C}_0(U,\omega) < \infty$ for open and bounded $U \subseteq T^*(\mathbb{T}^n)$ or $U \subseteq T^*(\mathbb{T}^n \times \mathbb{R}^m)$.

From [HZ12]: Let $(M,\ \omega)$ be a symplectic manifold with $\mathcal{C}_0(M,\ \omega) < \infty$, $H \in C^\infty(M)$ and

$$\overline{r}(H) \triangleq \left\{ c \in \mathbb{R}, \quad \begin{matrix} H^{-1}(c) \text{ nonempty and compact}, \\ dH(z) \neq 0 \text{ for all } z \in H^{-1}(c) \end{matrix} \right\}.$$

Then $H^{-1}(c)$ has a periodic orbit for Lebesgue a.e. $c \in \overline{r}(H)$.

Two Results of the Project

For two well-known **chaotic dynamical systems**[b], it is proved in [Li24] that

the **linked pendulum** system $(n \ge 2)$ admits a periodic orbit on *almost every* regular hypersurface[c], and

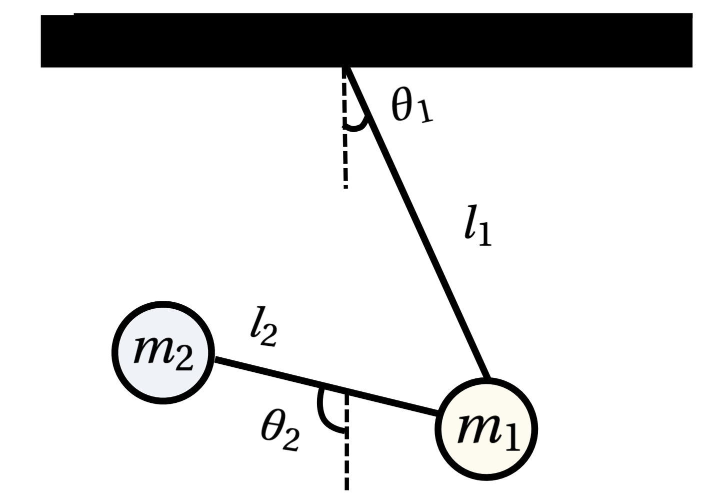
the **elastic pendulum** $(n \ge 2)$ admits a periodic orbit on *almost every* regular hypersurface.

[a] where $B(r) \triangleq \{z = (x,y) \in \mathbb{R}^{2n},\ |z|^2 < r^2\}$ and $Z(r) \triangleq \{z = (x,y) \in \mathbb{R}^{2n}, x_1^2 + y_1^2 < r^2\}$ for $r > 0$.

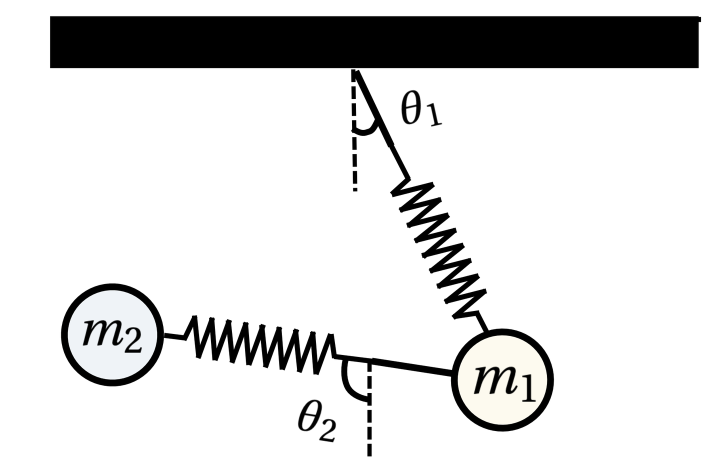
[b] 1) **sensitive to initial conditions** and 2) **trajectories are dense in the phase space** [Str18].

[c] A regular hypersurface $S \subseteq M$ is a codimension-1submanifold of M such that $S = H^{-1}(c)$ for some $H \in C^\infty(M)$, $c \in r(H)$ where

$$r(H) \triangleq \{c \in \mathbb{R}, \quad H^{-1}(c) \text{ nonempty}, \quad dH(z) \neq 0 \text{ for all } z \in H^{-1}(c) \quad \}.$$



Linked Pendulum



Elastic Pendulum

Remarks

Celestial Mechanics

$\mathcal{C}_0$ is unsuitable for systems **where the level sets of the Hamiltonian are not compact**. In particular: for the n-body system $(n \ge 2)$, we have $\overline{r}(H) = \varnothing$.[a]

Time-varying Systems

$\mathcal{C}_0$ is unsuitable for studying time-varying systems (whether **dissipative** or **with inputs**) due to its reliance on a specialized variational principle. This obstructs initial attempts to use $\mathcal{C}_0$ for **control Hamiltonian systems**.

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[a] The issue is the following: any **increase in kinetic energy** of the bodies can always be compensated by an equally large **decrease in the potential energy** (since $V(q) < 0$) by moving the bodies closer together.

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