The Optimal Reflection Problem

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Abstract

This work closely follows the paper in [YZZ20] concerning the channel estimation problem of intelligent reflective surfaces, and at times refers to recent results in complex-Hadamard matrices found within [TZ06]. We derive two user-receiver models and their equations for channel estimate errors. We then introduce the problem of optimal reflection, and following this, short — but hopefully delightful — expositions of each the various potential solutions and a terse discussion.

1. Introduction

This text is written for an audience at the undergraduate level in the applied sciences, with accompanying footnotes that aim to keep the working applied mathematician satisfied. We have deliberately opted for an algebraic exposition rather than an analytic one; and have chosen to avoid argumentation involving stochastic processes and the invertibility of Fourier Transform, et cetera.

The proper setting for the discussion herein is that of tensor[1] algebra. We will, however, not pursue this path — and we encourage the reader to consult [Rom07] for an abstract introduction to tensors and exterior algebra or [Tam09] for an introduction from an applied standpoint. Finally, the interested reader should peruse the now classic texts in [PP07; BLM12] for a rigorous introduction of the theory of tele- and digital communications.

2. Mathematical Preliminaries

In what follows, we will make no attempt at distinguishing scalar- or vector-valued quantities, as it should be clear from the context in which it is presented.

[1]A (k,l)-*tensor* on $\mathbb{R}^n$ is a $(k+l)$ multilinear mapping from $(\prod_{\underline{k}} \mathbb{R}^{n*}) \times (\prod_{\underline{l}} \mathbb{R}^n)$ into $\mathbb{R}$.

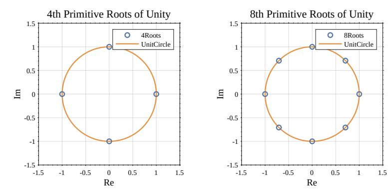

Figure 1: 4th and 8th Primitive Roots of Unity

Definition 2.1 Primitive kth root of unity

A *primitive kth root of unity* is a complex number $z \in S^1$[a] where k is the smallest positive integer such that $z^k = 1$. The set of all primitive kth roots of unity is $\left\{\exp(\frac{2\pi i}{k}), \exp(\frac{2\pi i}{k}2), \dots, 1\right\}$. See Figure 1.

[a]$S^1 = \{z \in \mathbb{C}^1, |z| = 1\}$ is the complex 1-sphere.

3. Telecommunications Primer

Note 3.1 Fourier Transforms

The Fourier Transform of x is denoted by $\mathcal{F}(x)$[a]. It is also convenient to identify x with its Fourier Transform.[b] If $y = h(x)(t)$[c] denotes an LTI system, an application of the Transform readily reads $y = \mathcal{H}x$. If $y = (y_{\underline{n}})$, $x = (x_{\underline{n}})$, and $h = (h_{\underline{n}})$ are vectoral, we also have $y = \mathcal{H}x$.

[a]Whenever it is defined in the classical or distributional sense.
[b]When $x \in L^2$, or in the sense of distributions.
[c]In the sense of distributions.

Definition 3.2 Channel

The *channel between two points* p_1, p_2 is a complex number denoted by the symbol — with round brackets included — $(p_1 \curvearrowright p_2)$. If p_1 wishes to *transmit* a signal χ to p_2, the *received signal* at p_2, denoted by R_χ is given by[a] $\mathrm{R}_\chi = (p_1 \frown p_2)\chi$.

[a]Under mild, but non-trivial mathematical regularity conditions, as discussed earlier.

Given that p_2 has exact information about the channel, the receipt R_χ allows for the recovery of

$$\chi = (p_1 \frown p_2)^{-1}\mathrm{R}_\chi.$$

Which motivates the general problem of *channel estimation.*

Channel Estimation: How do we estimate the channel between two (or more) points quickly and accurately?

Note 3.3 Equations for Channel Estimation

A common technique is to use a *pilot sequence*: a finite stream of complex numbers $(\chi_1, \dots \chi_n)$ that are a-priori known to p_1 and p_2. Let $\mathcal{H} = (p_1 \frown p_2)$, where in the presence of receiver noise[a] z_i during the transmission of the ith symbol, an easy computation shows that

$$\mathrm{R}_{\chi_i} = \mathcal{H}\chi_i + z_i \quad \text{assuming the channel fades slowly during pilot.}$$

Since i ranges through $\underline{n}$, we rewrite in matrix form: $(\mathrm{R}_{\chi_i})_{i=\underline{n}} = \mathrm{diag}_{i=\underline{n}}(\chi_i)\mathcal{H} + (z_i)_{i=\underline{n}}$.

Suppressing the dummy variable i, the receiver computes the *estimated channel* $\mathcal{H}_\mathrm{R} = \chi^{-1}(\mathrm{R}_\chi + z)$; whereas the *actual channel* is given by $\mathcal{H} = \chi^{-1}(\mathrm{R}_\chi)$. Hence, the squared 2-norm[b] of the *channel estimation error* is given by $\|\mathcal{H}_\mathrm{E}\|_2^2 = \|\chi^{-1}(z)\|_2^2 = \sigma^2 P^{-1}$ where σ^2 is the variance of z, and P is the power of p_1.

[a]z is assumed to be a wide-sense stationary L^2 process.
[b]$\|\cdot\|_p$ is the p-norm, deterministic or stochastic.

4. Optimal Reflection Problem

Definition 4.1 Intelligent Reflective Surface

An *intelligent reflective surface* is a n group of *(surface) elements* (ζ_j), each with a *(reflection) coefficient* ω_j which is a kth primitive root of unity[a].

[a]For simplicity, we assume ω_j is deterministic.

Let $(\chi_i)_{i=\underline{n+1}}$ be our pilot sequence, we write ω_{ij} as the coefficient of the ζ_j, during χ_i. One hopes that we can extract capacity by exploiting this added degree of flexibility. The main result by the authors of the paper in [YZZ20] concerns the *optimal reflection problem.*

Optimal Reflection Problem v1: Given n, k, concoct a recipe to optimize the *pilot reflection coefficients* ω_{ij} such that the channel estimation error $\|\mathcal{H}_E\|_2^2$ is kept at a minimum.

Note 4.2 Equations for Estimation of Vector of Channels with IRS

The transmitted signal from u to ζ_j at time i is simply the symbol χ_i, and the receipt at r during χ_i is a combination of

$$\mathrm{R}_{\chi_i} = \sum_{j=\underline{n}} \underbrace{(u \frown \zeta_j)(\zeta_j \frown r)}_{\text{reflective channel}}\chi_i + \underbrace{(u \frown r)'}_{\text{direct channel}}\chi_i + z_i,$$

where z_i is the noise with variance σ^2. After massaging the last expression by means of linear algebra, we have

$$\mathrm{R}_{\chi_i} = [1 \; (\omega_{i,j})_j] \underbrace{\begin{bmatrix} (u\frown r)' \\ (u\frown\zeta)(\zeta\frown r)^T \end{bmatrix}}_{=\mathcal{H}} \chi_i + z_i, \qquad (1)$$

where j is taken through $\underline{n}$. $\mathcal{H}$ refers to the *vector of channels* when $\omega_{i,j} \equiv 1$. Let us relabel $\Omega(i) = (1, (\omega_{ij})_{j=\underline{n}}) = (1, \omega_{i1}, \dots, \omega_{in})$, and Ω is the $(n+1)$-matrix whose ith row is $\Omega(i)$. By combining $\chi_{\underline{n+1}}$, we obtain

$$\mathrm{R}_\chi = \chi\Omega\mathcal{H} + z \quad \text{and} \quad (\Omega\chi)^{-1}(\mathrm{R}_\chi - z) = \mathcal{H}.$$

The receiver cannot distinguish between $\chi\Omega\mathcal{H}$ and z, as they compute $\mathcal{H}_R = (\Omega\chi)^{-1}(\mathrm{R}_\chi)$. The *channel estimation error* is therefore $\mathcal{H}_E = \mathcal{H} - \mathcal{H}_R$ — whose squared 2-norm takes on a distinguished form:

$$\|\mathcal{H}_E\|_2^2 = \|\Omega^{-1}\|_\mathbf{F}^2 \|\chi^{-1}z\|_2^2 = \sigma^2 P^{-1} \|\Omega^{-1}\|_\mathbf{F}^2 \text{[a]}.$$

[a]$\|\cdot\|_\mathbf{F}$ is the *Frobenius norm*, where $\|A\|_\mathbf{F}^2 = \mathrm{trace}(AA^*)$ with A^* denoting the *conjugate transpose* of A.

5. Hadamard Matrices

Our goal now is to infimize $\|\Omega^{-1}\|_\mathbf{F}^2$, such that each ω_{ij} is a primitive kth root of unity,

$$\Omega = \begin{bmatrix} 1 & \omega_{1,1} & \cdots & \omega_{1,n+1} \\ 1 & \vdots & \vdots & \vdots \\ 1 & \vdots & \vdots & \vdots \\ 1 & \omega_{n+1,1} & \cdots & \omega_{n+1,n+1} \end{bmatrix} \quad \text{and } \Omega \in GL(n+1).$$

Given n, k, it is clear that a minimizing solution exists. We temporarily relax the requirement that ω_{ij} is a primitive kth root of unity, and present two important types of matrices.

Definition 5.1 Real-Hadamard Matrix

A real n-matrix $A = (a_{ij})$ is *Hadamard* whenever $a_{ij} = \pm 1$, and $AA^T = n\,\mathrm{id}_{\mathbb{R}^n}$.

Definition 5.2 Complex-Hadamard Matrix

A complex n-matrix $A = (a_{ij})$ is *Hadamard* whenever $a_{ij} \in S^1$, and $AA^* = n\,\mathrm{id}_{\mathbb{C}^n}$.

It would turn out that the Hadamard matrices (if they exist) are the matrices that minimize $\|\Omega^{-1}\|_{\mathbf{F}}^2$. The following Proposition holds given any n, k, with the obvious inference that $\mathbb{R} \subseteq \mathbb{C}$.

Proposition 5.3 Hadamard matrices minimize L2 Norms

Every complex-Hadamard n-matrix that is an admissable solution is minimizing, and conversely: every admissable minimizing solution is complex-Hadamard.

Proof. The paper in [YZZ20] contains a sketch of such a proof. ∎

With this, we restate the goal of the paper.

> **Optimal Reflection Problem v2:** If $k \geq 3$, then find a complex n-Hadamard matrix whose entries are kth primtive roots of unity; and when $k = 2$: find a real n-Hadamard matrix.

5.1. Real-Hadamard Matrices

In the case of real-Hadamard matrices, it is still an open question[2] whether or not a real-Hadamard matrix exists for an arbitrary $n \geq 1$. However, if n is special, then the existence of such a matrix follows from famous result (which we will prove!) from matrix algebra. First, we must introduce a matrix operation.

Definition 5.4 Kronecker Product

Let $A = (a_{ij})$ be a real matrix, and $B \in \mathbb{R}^{p \times q}$. The *Kronecker Product between A and B* — denoted by $A \otimes B$[a] — is found by replacing each of the scalar entries a_{ij} of A with the block matrix $a_{ij}B \in \mathbb{R}^{p\times q}$. [Tam09]

[a] The command in Matlab for $A \otimes B$ is `kron(A,B)`.

Example 5.5 Symplectic Form

The *standard symplectic form* on $\mathbb{R}^{2n}$ can be constructed by $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \otimes \mathrm{id}_{\mathbb{R}^n} = \begin{bmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ -\mathrm{id}_{\mathbb{R}^n} & 0 \end{bmatrix}$.

Proposition 5.6 Sylvester's Construction

Let $n = 2^m$ where $m = 0, 1, \ldots,$ there exists a real-Hadamard n-matrix; and it is given by $H_{2^m} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right] \otimes H_{2^{m-1}}$, where $H_1 = \mathrm{id}_{\mathbb{R}}$.

Proof. By inducting on m, we set $A = \left[\begin{smallmatrix} H & H \\ H & -H \end{smallmatrix}\right]$, where $HH^T = n\,\mathrm{id}_{\mathbb{R}^n}$. We wish to show that $AA^T = (2n)\,\mathrm{id}_{\mathbb{R}^{2n}}$; to wit,

$$AA^T = \begin{bmatrix} H & H \\ H & -H \end{bmatrix}\begin{bmatrix} H^T & H^T \\ H^T & -H^T \end{bmatrix} = \begin{bmatrix} HH^T + HH^T & HH^T - HH^T \\ HH^T - HH^T & (-H)(-H)^T + (-H)(-H)^T \end{bmatrix}$$
$$= \begin{bmatrix} (2n)\,\mathrm{id}_{\mathbb{R}^n} & 0 \\ 0 & (2n)\,\mathrm{id}_{\mathbb{R}^n} \end{bmatrix} = (2n)\,\mathrm{id}_{\mathbb{R}^{2n}}, \tag{2}$$

and therefore A is Hadamard. ∎

Example 5.7 Sylvester for $m = 1, 2$

Matlab generates real H_n for $n = 2^m$, among other n[a]. We see that

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{which gives birth to} \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

[a] Using the Matlab command `hadamard(n)`.

Remark 5.8 Brute-force = Hopeless

Any brute-force attempt towards a real-Hadamard matrix for $n \notin \{2^m, 12 \times 2^m, 20 \times 2^m\}$ is *computationally intractable* even for moderately large n — for example, H_{668} still eludes us. [TZ06; YZZ20]

5.2. Complex-Hadamard Matrices

It is well known among image and signal processing circles that complex-Hadamard matrices exist for all n, even better: their entries are *precisely* the primitive roots of unity. We introduce a distinguished subset of them to the reader in Example 5.9.

Example 5.9 DFT Matrices and Rescaled DFT Matrices

In general, the nth *Discrete Fourier Transform* matrix is $\mathrm{DFT}(n) = n^{-1/2}[z^{(j-1)(k-1)}]$, where $j, k = \underline{n}$, and $z = \exp(2\pi i n^{-1})$.[a]

The *rescaled DFT matrices* are defined by removing the factor of $n^{-1/2}$, that is: $\mathrm{RDFT}(n) = n^{1/2}\mathrm{DFT}(n)$.

[a] The nth DFT matrix is given by `dftmtx(n)` in Matlab.

Proposition 5.10 Rescaled DFT matrices are complex-Hadamard

The RDFT matrices, as described in Example 5.9, are complex-Hadamard for every $n \geq 1$.

Proof. Follows from Proposition 5.3. ∎

[2] To be more precise, it is shown that if n is not divisible by 4, and $n \geq 4$, then no real n-Hadamard matrices can exist. [TZ06] We must also emphasize that does not imply the existence of H_n whenever $n = 4m$.

Note 5.11 Problems with Rescaled DFT Matrices
An influential paper on the physical design of IRS suggests that $k \in \{4,8\}$ provides the best engineering results, and increasing k beyond 2^4 provides negligible increases in SNR. [Wu+08] We have also thus far assumed that each $\omega_{i,j}$ is deterministic — a well known result in telecommunications tells us that a stochastic phase shift[a] broadens the spectrum. And finally, a typical IRS will have hundreds if not thousands of elements, and hence $k \ll n$.

[a] Brought about by complexity.

Remark 5.12 Sylvester's Construction doubles any Hadamard matrix.
Proposition 5.6 applies for an arbitrary real- or complex-Hadamard matrix H. In addition to H_1, known real-*generators* of H_n[a] are H_{12}, H_{20}[b].

[a] Whose existence is still unproven.
[b] Not exhaustive.

6. Suboptimal Reflection Patterns

In view of these difficulties, and given that the pursuit of arbitrary admissable complex-Hadamard matrices seems to be (to our knowledge) completely hopeless[TZ06], let us find an approximate solution.

> **Optimal Reflection Problem v3:** Given $n \geq 1$, find a basis Ω consisting of entries ± 1 such that Ω is as orthogonal as possible.

The authors in [YZZ20] proposed to cut a piece of a larger, real-Hadamard matrix — and ran simulations that show that, in some cases, is better some other heuristics.

Definition 6.1 Truncated Hadamard matrix
Let $n \geq 1$, a matrix H_n^p is called a *truncated Hadamard matrix of size n* whenever H_n^p is a cropped version of a real p-Hadamard matrix H_p, where $p \geq n$.

H_n^p is said to be *minimal* whenever p is chosen to be the least of such p.[a].

[a] Such a p exists by Proposition 5.6.

Example 6.2 A fake H_7 from a truncated H_8
A real 7-Hadamard matrix does not exist, yet we can *approximate* one by truncation. We show one such H_7^8 below.

$$H_8 = \left[\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ \hline 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right] = \left[\begin{array}{c|c} H_7^8 & \vdots \\ \hline \cdots & * \end{array}\right].$$

7. Discussion and Concluding Remarks

For the results of their numerical simulations and comparisons with other techniques, the authors refer the reader to the original paper [YZZ20]. We conclude this report with a short discussion.

First, the method of truncating known Hadamard matrices seems almost trivial at first, but the authors (of this report) hope that we have presented sufficient motivation for this.

Second, the simulation laid bare in [YZZ20] does not include the Hadamard matrices generated from the *known* families $\{H_{12}, H_{20}\} \cup \{\text{RDFT}(n)\}_{n \geq 1}$.

Third, the simulation results provided only include small n, up to $n \leq 40$ groups or $n \leq 200$ elements. Fourth, it is unknown whether or not the proposed algorithm, in the second half of the paper (which is beyond the scope of this report) is viable outside of a single-user setting.

Fifth, it must be stated that the study of *almost Hadamard matrices*, and *circulant matrices* are active areas of mathematical research. By looking at geometries such as $p = 1$, fruitful improvements over the truncation technique may be found. [BN13].

Finally, it is our hope that this report stands as a technically sound, rigorously structured document that balances mathematical depth with clarity, and is a valuable resource for its intended audience.

8. Acknowledgements

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